THE PROBLEM OF DETERMINING THERMAL RESISTANCE*

D. I. Zaks

UDC 536.2

It is demonstrated that the thermal resistance of bodies exhibiting conductive thermal conductivity depends on the form of the temperature field at the surfaces [S_1 and S_2] bounding the body. This relationship has been worked out on a model problem.

The overheating of a body is frequently characterized by the thermal resistance R, determined [1] from the formula

$$R=\frac{\theta_1-\theta_2}{P}. \tag{1}$$

However, in real designs the surfaces S_1 and S_2 are generally not isothermal. In calculation practice [2] relationship (1) is therefore replaced by

$$\bar{R}=\frac{\bar{\theta}_1-\bar{\theta}_2}{P}. \tag{2}$$

Here it is assumed that (1) and (2) are identical.

Let us examine this problem in greater detail. First of all, we note that

$$\bar{R}>R. \tag{3}$$

There is hardly any point in presenting the proof of this inequality, since an inequality similar to (3) for inverse electrical capacitances C^{-1} is well known in electrical engineering [3]. Using the theory of similarity to replace C^{-1} by the thermal resistances, we derive inequality (3).

To relate $\bar{R}$ to R, let us examine a medium v_2 bounded by surfaces S_1 and S_2 (see Fig. 1a). We find $\bar{R}$ and R between S_1 and S_2. For this, we choose the functions ψ and θ expressed in units of temperature and we subject these to the conditions

$$\Delta\psi=0,\quad \Delta\theta=0,\quad (x,\ y,\ z)\in v_2-S_1,\ S_2; \tag{4a}$$

$$\psi=PR,\quad \theta=\theta_1(x,\ y,\ z),\quad (x,\ y,\ z)\in S_1; \tag{4b}$$

$$\psi=0,\quad \theta=\theta_2(x,\ y,\ z),\quad (x,\ y,\ z)\in S_2; \tag{4c}$$

$$\int\limits_{S_1}\frac{\partial\theta}{\partial n}dS=-\frac{P}{\lambda_2}. \tag{4d}$$

So as to be able to use the theorem of the average in the future, we will also require that the function θ does not change sign over its entire region of existence. This condition is easily satisfied by appropriate location of the reckoning origin for the temperature.

As we can see from (4), the function ψ is isothermal at the surfaces S_1, and S_2, while the function θ has been specified arbitrarily at S_1 and S_2.

We will use the Green's formula [4] in the region of v_2

$$\int\limits_{S_1+S_2}\left(\psi\frac{\partial\theta}{\partial n}-\theta\frac{\partial\psi}{\partial n}\right)dS=\int\limits_{v_2}(\psi\Delta\theta-\theta\Delta\psi)\,dv. \tag{5}$$

With consideration of (4), Eq. (5) is transformed to

*In this paper we will be talking exclusively of the processes of conduction heat transfer.

Radio Engineering Institute, Tagrarog. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 4, pp. 742-745, April, 1969. Original article submitted April 8, 1968.

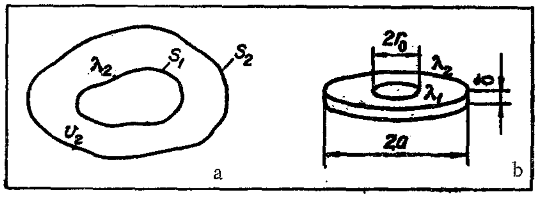

Fig. 1. Theoretical model (a) for the derivation of relationships (1)-(7) and disk (b) in an unbounded medium (model problem).

$$\int_{S_1}\left(\psi\frac{\partial\theta}{\partial n}-\theta\frac{\partial\psi}{\partial n}\right)dS=\int_{S_2}\theta\frac{\partial\psi}{\partial n}dS.$$

Using the theorem of the average and condition (4d), we find

$$-\frac{P^2R}{\lambda_2}-\frac{\overline{\partial\psi}}{\partial n}(S_1)\bar{\theta}_1S_1=\frac{\overline{\partial\psi}}{\partial n}(S_2)\bar{\theta}_2S_2, \tag{6}$$

where

$$\frac{\overline{\partial\psi}}{\partial n}(S_i)=\int_{S_i}\theta\frac{\partial\psi}{\partial n}dS\Big/\int_{S_i}\theta dS.$$

Further transformations lead to

$$R=-\frac{\lambda_2}{P}\left\{\frac{\overline{\partial\psi}}{\partial n}(S_1)S_1\bar{R}+\frac{\bar{\theta}_2}{P}\left[\frac{\overline{\partial\psi}}{\partial n}(S_1)S_1+\frac{\overline{\partial\psi}}{\partial n}(S_2)S_2\right]\right\}. \tag{7}$$

Formula (7) relates the thermal resistances given by (1) and (2).

We examined a model problem to achieve a quantitative evaluation of (7). A thin disk with a coaxial heat source of constant power served as our model with the heat source placed into an infinite medium exhibiting a different thermal conductivity (see(Fig. 1b). The mathematical formulation of this problem is of the following form:

within the disk

$$\Delta\theta^{\mathrm{I}}=\begin{Bmatrix}-\dfrac{q}{\lambda_1} & 0\leqslant r\leqslant r_0\\ 0 & r_0<r\leqslant a\end{Bmatrix},$$

outside the disk

$$\Delta\theta^{\mathrm{II}}=0,$$

at the surface of the disk

$$\theta^{\mathrm{I}}=\theta^{\mathrm{II}},\quad \lambda_1\frac{\partial\theta^{\mathrm{I}}}{\partial n}=\lambda_2\frac{\partial\theta^{\mathrm{II}}}{\partial n}.$$

Solving this problem by the method of successive approximations, we find θ^{I} and $\bar{R}=S_1^{-1}P^{-1}\int_{S_1}\theta^{\mathrm{I}}dS$.

Knowing $R = 1/8a\lambda_2$ for the isothermal disk [5], we finally find

$$\frac{\bar{R}}{R}\approx 1+\frac{4a\lambda_2}{\pi\delta\lambda_1}\left[-0.053+\frac{a^2}{3r_0^2}\left(1-\sqrt{1-r_0^2a^{-2}}\right)-\frac{2}{3}\sqrt{1-r_0^2a^{-2}}-\frac{r_0^2}{4a^2}+\ln\left(1+\sqrt{1-r_0^2a^{-2}}\right)\right]. \tag{8}$$

Relationship (8) is satisfied under the conditions

$$\frac{a}{\delta}\gg 1,\quad \frac{\lambda_1}{\lambda_2}\gg 1,\quad \delta\lambda_1|_{\delta\to 0}=\mathrm{const}. \tag{9}$$

If we assume $\lambda_1 = \lambda_2$ and $\delta = 0$, we can find the exact solution for the model problem:

$$\frac{\bar{R}}{R}=\frac{4}{\pi}\qquad \text{for}\ \ r_0=0; \tag{10}$$

$$\frac{\bar{R}}{R}=0.8\,\frac{4}{\pi}\qquad \text{for}\ \ r_0=a. \tag{11}$$

The results from the calculation of $\bar{R}/R$ with (9), (10), and (11) are shown in Table 1.

As we can see from the Table 1, when $r_0 = a$ we have $\bar{R} \approx R$. The condition $r_0 = a$ indicates that the heat source is uniformly distributed over the entire surface S_1. Problems with such a boundary condition are usually solved more easily than with the condition of isothermicity at S_1.

Thus, in calculating the thermal resistance of bodies exhibiting conductive thermal conductivity we are confronted with a certain indeterminacy. This is a consequence of the fact that the relief of the temperature field at S_1 and S_2 for P = const may differ for identical structures. To eliminate this indeterminacy, we must stipulate the nature of the temperature field at the surfaces S_1 and S_2.

It is apparently advisable to base the determination of the thermal resistance on (1).

TABLE 1. The Ratio of $\bar{R}/R$ as a Function of the Disk Parameters

$\frac{r_0}{a}$	0	0,5	1,0	Parameter values
$\frac{\bar{R}}{R}$	1,27	—	1,02	$\frac{\lambda_1}{\lambda_2}=1,\ \delta=0$
$\frac{\bar{R}}{R}$	1,15	1,12	1,03	$\frac{4a\lambda_2}{\pi\delta\lambda_1}=1$

NOTATION

θ_1 and θ_2 are constant temperatures at the surfaces S_1 and S_2;

S_1 and S_2 are the surfaces between which the thermal resistance is determined;

P is the heat flux (power) passing through the surfaces S_1 and S_2;

$\bar{\theta}_i = S_i^{-1}\int_{S_i}\theta\, dS$ denotes the surface-averaged temperatures at S_i;

a and δ are the radius and thickness of the disk;

r_0 and δ are the radius and thickness of the heat source;

λ_1 and λ_2 are the coefficients of thermal conductivity for the disk and for the ambient medium v_2;

q is the density of the heat source;

θ^I and θ^{II} are the temperatures inside and outside the disk.

LITERATURE CITED

1. P. Schneider, Engineering Problems in Heat Conduction [Russian translation], IL (1960).
2. N. M. Dul'neva, Author's Abstract of Candidate's Dissertation [in Russian], LPI im. Kalinina (1966).
3. G. Polya and G. Seryo, Isoperimetric Inequalities in Mathematical Physics [Russian translation], Fizmatgiz (1962).
4. N. S. Koshlyakov, É. B. Gliner, and M. M. Smirnov, Basic Differential Equations of Mathematical Physics [in Russian], Fizmatgiz (1962).
5. N. N. Lebedev, I. P. Skal'skaya, and Ya. S. Uflyand, Collection of Problems in Mathematical Physics [in Russian], GITTL (1965).